Computational Challenges in Perfect form theory

Mathieu Dutour Sikirić
Rudjer Bošković Institute, Zagreb Croatia

April 24, 2018
I. Enumerating Perfect forms
Notations

- We define $S^n$ the space of symmetric matrices, $S^n_{>0}$ the cone of positive definite matrices.
- For $A \in S^n_{>0}$ define $A[x] = x A x^T$,
  \[
  \min(A) = \min_{x \in \mathbb{Z}^n - \{0\}} A[x] \text{ and } \text{Min}(A) = \{x \in \mathbb{Z}^n \text{ s.t. } A[x] = \min(A)\}
  \]
- A matrix $A \in S^n_{>0}$ is perfect (Korkine & Zolotarev) if the equation
  \[
  B \in S^n \text{ and } B[x] = \min(A) \text{ for all } x \in \text{Min}(A)
  \]
  implies $B = A$.
- If $A$ is perfect, then its perfect domain is the polyhedral cone
  \[
  \text{Dom}(A) = \sum_{v \in \text{Min}(A)} \mathbb{R}_+ p(v).
  \]
- The Ryshkov polyhedron $\mathcal{R}_n$ is defined as
  \[
  \mathcal{R}_n = \{A \in S^n \text{ s.t. } A[x] \geq 1 \text{ for all } x \in \mathbb{Z}^n - \{0\}\}$
Known results on perfect form enumeration

<table>
<thead>
<tr>
<th>dim.</th>
<th>Nr. of perfect forms</th>
<th>Best lattice packing</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1 (Lagrange)</td>
<td>A₂</td>
</tr>
<tr>
<td>3</td>
<td>1 (Gauss)</td>
<td>A₃</td>
</tr>
<tr>
<td>4</td>
<td>2 (Korkine &amp; Zolotarev)</td>
<td>D₄</td>
</tr>
<tr>
<td>5</td>
<td>3 (Korkine &amp; Zolotarev)</td>
<td>D₅</td>
</tr>
<tr>
<td>6</td>
<td>7 (Barnes)</td>
<td>E₆ (Blichfeldt &amp; Watson)</td>
</tr>
<tr>
<td>7</td>
<td>33 (Jaquet)</td>
<td>E₇ (Blichfeldt &amp; Watson)</td>
</tr>
<tr>
<td>8</td>
<td>10916 (DSV)</td>
<td>E₈ (Blichfeldt &amp; Watson)</td>
</tr>
<tr>
<td>9</td>
<td>≥ 9,200,000</td>
<td>Λ₉?</td>
</tr>
</tbody>
</table>

- The enumeration of perfect forms is done with the Voronoi algorithm.
- Blichfeldt used Korkine-Zolotarev reduction theory.
- Perfect form theory has applications in
  - Lattice theory for the lattice packing problem.
  - Computation of homology groups of $GL_n(\mathbb{Z})$.
  - Compactification of Abelian Varieties.
Perfect forms in dimension 9

- Finding the perfect forms in dimension 9 would solve the lattice packing problem.

- Several authors did partial enumeration of perfect forms in dimension 9:
  - Schürmann & Vallentin: $\geq 500000$
  - Anzin: $\geq 524000$
  - Andreanov & Scardicchio: $\geq 500000$ (but actually $1.10^6$)
  - van Woerden: $\geq 9.10^6$

So, one does not necessarily expect an impossibly large number.

- Other reason why it may work:
  - Maximal kissing number is 136 (by Watson)
  - The number of complex cones (with number of rays greater than $n(n + 1)/2 + 20$) is not too high.
  - Many cones have a pyramid decomposition: $C = C' + \mathbb{R}_+ v_1 + \cdots + \mathbb{R}_+ v_r$ with $\dim C' = \dim C - r$
Needed tools: Canonical form

- The large number of perfect forms mean that we need special methods for isomorphism
- Two alternatives:
  - Use very fine invariants: \((\det(A), \min(A))\) is already quite powerful.
  - Use a canonical form.
- Minkowski reduction provides a canonical form but is hard to compute.
- Isomorphism and stabilizer computations can be done by ISOM/AUTOM but we risk being very slow if the invariant are not fine enough.
- Partition backtrack programs for graph isomorphism (nauty, bliss, saucy, traces, etc.) provides a canonical form for graphs.
- Using this we can:
  - Find a canonical form for edge weighted graphs.
  - Find a canonical ordering of the shortest vectors.
  - Find a canonical presentation of the shortest vectors.
  - Find a canonical representation of the form.
Needed tools: MPI parallelization and Dual description

- There are two essential difficulties for the computation:
  - The very large number of perfect forms.
  - The difficult to compute perfect forms whose number of shortest vectors is very high.

- For the first problem, the solution is to use MPI (Message Passing Interface) formalism for parallel computation. This can scale to thousands of processors (work with Wessel van Woerden).

- The dual description problem is harder:
  - As mentioned, many cones are pyramid and thus their dual description is relatively easy.
  - But many cones, in particular the one of \( \Lambda_9 \), are not so simple but yet have symmetries.
  - We need to use symmetries for this computation. The methods exist.
  - The critical problem is that we need a permutation group library in C++.
II. Well rounded retract and homology
Well rounded forms and retract

- A form $Q$ is said to be well rounded if it admits vectors $v_1, \ldots, v_n$ such that
  - $(v_1, \ldots, v_n)$ form a $\mathbb{R}$-basis of $\mathbb{R}^n$ (not necessarily a $\mathbb{Z}$-basis)
  - $v_1, \ldots, v_n$ are shortest vectors of $Q$.
- Such vector configurations correspond to bounded faces of $\mathcal{R}_n$.
- Every form in $\mathcal{R}_n$ can be continuously deformed to a well rounded form and this defines a contractible polyhedral complex $\mathcal{WR}_n$ of dimension $\frac{n(n-1)}{2}$.
- Every face of $\mathcal{WR}_n$ has finite stabilizer.
- $\mathcal{WR}_n$ is essentially optimal (Pettet, Souto, 2008).
Topological applications

- The fact that $\mathcal{WR}_n$ is contractible, has finite stabilizers, and is acted on by $\text{GL}_n(\mathbb{Z})$ means that we can compute rational homology of $\text{GL}_n(\mathbb{Z})$.
- This has been done for $n \leq 7$ (Elbaz-Vincent, Gangl, Soulé, 2013).
- We can get $K_8(\mathbb{Z})$ (DS, Elbaz-Vincent, Martinet, in preparation).
- By using perfect domains, we can compute the action of Hecke operators on the cohomology.
- This has been done for $n \leq 4$ (Gunnells, 2000).
- Using $T$-space theory this can be extended to the case of $\text{GL}_n(R)$ with $R$ a ring of algebraic integers:
  - For Eisenstein and Gaussian integers, this means using matrices invariant under a group.
  - For other number fields with $r$ real embeddings and $s$ complex embeddings this gives a space of dimension
    $$r \frac{n(n + 1)}{2} + sn^2$$
III. Tesselations: Central cone compactification

or

?
Linear Reduction theories in $S_{\geq 0}^n$

Decompositions related to perfect forms:
- The perfect form theory (Voronoi I) for lattice packings (full face lattice known for $n \leq 7$, perfect domains known for $n \leq 8$)
- The central cone compactification (Igusa & Namikawa) (Known for $n \leq 6$)

Decompositions related to Delaunay polytopes:
- The $L$-type reduction theory (Voronoi II) for Delaunay tessellations (Known for $n \leq 5$)
- The $C$-type reduction theory (Ryshkov & Baranovski) for edges of Delaunay tessellations (Known for $n \leq 5$)

Fundamental domain constructions:
- The Minkowski reduction theory (Minkowski) it uses the successive minima of a lattice to reduce it (Known for $n \leq 7$) not face-to-face
- Venkov’s reduction theory also known as Igusa’s fundamental cone (finiteness proved by Venkov and Crisalli)
Central cone compactification

We consider the space of integral valued quadratic forms:

\[ \mathcal{I}_n = \{ A \in S^n_{>0} \text{ s.t. } A[x] \in \mathbb{Z} \text{ for all } x \in \mathbb{Z}^n \} \]

All the forms in \( \mathcal{I}_n \) have integral coefficients on the diagonal and half integral outside of it.

The centrally perfect forms are the elements of \( \mathcal{I}_n \) that are vertices of \( \text{conv} \mathcal{I}_n \).

For \( A \in \mathcal{I}_n \) we have \( A[x] \geq 1 \). So, \( \mathcal{I}_n \subset \mathcal{R}_n \).

Any root lattice gives a vertex both of \( \mathcal{R}_n \) and \( \text{conv} \mathcal{I}_n \).

The centrally perfect forms are known for \( n \leq 6 \):

<table>
<thead>
<tr>
<th>dim.</th>
<th>Centrally perfect forms</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( A_2 ) (Igusa, 1967)</td>
</tr>
<tr>
<td>3</td>
<td>( A_3 ) (Igusa, 1967)</td>
</tr>
<tr>
<td>4</td>
<td>( A_4, D_4 ) (Igusa, 1967)</td>
</tr>
<tr>
<td>5</td>
<td>( A_5, D_5 ) (Namikawa, 1976)</td>
</tr>
<tr>
<td>6</td>
<td>( A_6, D_6, E_6 ) (DS)</td>
</tr>
</tbody>
</table>

By taking the dual we get tessellations in \( S^n_{\geq 0} \).
Suppose that we have a conjecturally correct list of centrally perfect forms $A_1, \ldots, A_m$. Suppose further that for each form $A_i$ we have a conjectural list of neighbors $N(A_i)$.

We form the cone

$$C(A_i) = \{X - A_i \text{ for } X \in N(A_i)\}$$

and we compute the orbits of facets of $C(A_i)$.

For each orbit of facet of representative $f$ we form the corresponding linear form $f$ and solve the Integer Linear Problem:

$$f_{opt} = \min_{X \in I_n} f(X)$$

It is solved iteratively (using glpk) since $I_n$ is defined by an infinity of inequalities.

If $f_{opt} = f(A_i)$ always then the list is correct. If not then the $X$ realizing $f(X) < f(A_i)$ need to be added to the full list.
IV. Perfect coverings
Problem setting and algorithm

We have a $d$ dimensional cone $C$ embedded into $S^n_{>0}$ and we want to find a set of perfect matrix $A_1, \ldots, A_m$ such that

$$C \subset Dom(A_1) \cup \cdots \cup Dom(A_m)$$

We want the cones having an intersection that is full dimensional in $C$ (this is for application in Algebraic Geometry).

We take a cone $C$ in $S^n_{>0}$ of symmetry group $G$.

- We start by taking a matrix $A$ in the interior of $C$.
- We compute a perfect form $B$ such that $A \in Dom(B)$ and insert $B$ into the list of orbit
- We iterate the following:
  - For each untreated orbit of perfect domain in $O$ compute the facets.
  - For each facet do the flipping and keep if the intersection with $C$ is full dimensional in $C$.
  - Insert the obtained perfect domains if they are not equivalent to a known one.
The space intersection problem

- Given a family of vectors \((v_i)_{1 \leq i \leq M}\) spanning a cone \(C \in \mathbb{R}^n\) and a \(d\)-dimensional vector space \(S\) we want to compute the intersection \(C \cap S\) that is facets and/or extreme rays description.
- In the case considered we have \(d\) small.
- Tools:
  - We can compute the group of transformations preserving \(C\) and \(S\).
  - We can check if a point in \(S\) belongs to \(C \cap S\) by linear programming.
  - We can test if a linear inequality \(f(x) \geq 0\) defines a facet of \(C \cap S\) by linear programming.
- Algorithm:
  - Compute an initial set of extreme rays by linear programming.
  - Compute the dual description using the symmetries.
  - For each facet found, check if they are really facet. If not add the missed extreme rays and iterate.
V. Perfect domains for symplectic group
The invariant manifold

- We are interested in the group $G = \text{Sp}(2n, \mathbb{Z})$ defined as

$$G = \left\{ M \in \text{GL}_{2n}(\mathbb{Z}) \text{ s.t. } MJM^T = J \right\} \text{ with } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

- We want something similar to $\text{GL}_n(\mathbb{Z})$.

- The idea is to introduce the manifold

$$M_G = \left\{ MM^T \text{ for } M \in \text{Sp}(2n, \mathbb{Z}) \right\} = \left\{ A \in S_{>0}^n \text{ s.t. } AJA^T = J \right\}$$

- We have $M_G \subset S_{>0}^n$, $G$ acts on it and it is contractible.

- We can consider the perfect domains $\text{Dom}(A)$ that intersects $M_G$ in their interior.

- The number of orbits of such perfect domains under $\text{Sp}(2n, \mathbb{Z})$ ought to be finite.

- Maybe the method also applies to other groups $G \subset \text{GL}_n(\mathbb{Z})$?
Results for $n = 2$ and $n = 3$

- In *MacPherson & M. McConnell, 1993* a reduction theory for $\text{Sp}(4, \mathbb{Z})$ is given:
  - It describes a cell complex on which $G$ acts.
  - Number of orbits in the decomposition are 1(vector), 1(Lagrangian space), 2, 3, 3, 2, 2($A_4$ or $D_4$).
  - The rank in the decomposition does not correspond to the linear algebra rank.

- We can effectively compute the homology and Hecke operators for $n = 2$ (Joint work with P. Gunnells). So far we have
  - Computed 4 Hecke operators
  - Computed for the Siegel subgroups of $\text{Sp}(4, \mathbb{Z})$ up to $p = 19$.
Need more optimization and computational power.

- For $n = 3$ likely there is a similar decomposition. We took at random points in the manifold and computed the corresponding perfect domain and obtained 22 orbits so far.
VI. Perfect form complex
Perfect form complex

- Each orbit of face corresponds to a vector configuration.
- The rank $rk(\mathcal{V})$ of a vector configuration $\mathcal{V} = \{v_1, \ldots, v_m\}$ is the rank of the matrix family $\{p(v_i) = v_i^T v_i\}$.
- The complex is fully known for $n \leq 7$. Number of orbits by rank (Elbaz-Vincent, Gangl, Soulé, 2013):
  - $n = 4$: 1, 3, 4, 4, 2, 2, 2
  - $n = 5$: 2, 5, 10, 16, 23, 25, 23, 16, 9, 4, 3
  - $n = 6$: 3, 10, 28, 71, 162, 329, 589, 874, 1066, 1039, 775, 425, 181, 57, 18, 7
  - $n = 7$: 6, 28, 115, 467, 1882, 7375, 26885, 87400, 244029, 569568, 1089356, 1683368, 2075982, 2017914, 1523376, 876385, 374826, 115411, 24623, 3518, 352, 33
- It is out of question to enumerate the whole perfect form complex in dimension 8.
- Instead the idea is to try to enumerate the cells in lowest rank and go upward in rank.
Testing realizability of vector families I

- **Problem:** Suppose we have a configuration of vector $\mathcal{V}$. Does there exist a matrix $A \in S_{>0}^n$ such that $\text{Min}(A) = \mathcal{V}$?

- Consider the linear program

$$\begin{align*}
\text{minimize} & \quad \lambda \\
\text{with} & \quad \lambda = A[v] \text{ for } v \in \mathcal{V} \\
& \quad A[v] \geq 1 \text{ for } v \in \mathbb{Z}^n - \{0\} - \mathcal{V}
\end{align*}$$

If $\lambda_{opt} < 1$ then $\mathcal{V}$ is realizable, otherwise no.

- In practice one replaces $\mathbb{Z}^n$ by a finite set $\mathcal{Z}$ and iteratively increases it until a conclusion is reached.

- A related problem is to find the smallest configuration $\mathcal{W}$ such that there exist a $A \in S_{>0}^n$ with $\mathcal{V} \subseteq \mathcal{W} = \text{Min}(A)$ and possibly $\text{rk}(\mathcal{V}) = \text{rk}(\mathcal{W})$.

- The major problem is to limit the number of iterations.
A vector configuration can be simplified by applying LLL reduction to the positive definite quadratic form $\sum_i v_i v_i^T$. This diminishes the coefficient size.

We can use the $\text{GL}_n(\mathbb{Z})$-symmetries of $\mathcal{V}$ to diminish the size of the problem.

The linear programs occurring are potentially very complex. We need exact solution fast technique for them. The idea is to use double precision and $\text{glpk}$. From this we search for a primal/dual solution. If failing we use the simplex method in rational arithmetic with $\text{cdd}$.

According to the optimal solution $A_0$:

- If $A_0$ is positive definite but there is a $v$ such that $A_0[v] < 1$ then insert it into $\mathcal{Z}$.
- If $\text{Ker}(A_0) \neq 0$ we take a $v$ with $A_0 v = 0$ and insert it into $\mathcal{Z}$.
- If $A_0$ is not positive semidefinite, we take an eigenvector of negative eigenvalue and search for rational approximation $v$. 
Simpliciality results

- The perfect form complex provides a compactification of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties, which is a canonical model in the sense of the minimal model program (Shepherd-Barron, 2006).
- For a cone in the perfect form complex, we can consider if it is simplicial or if it is basic, i.e. if its generators can be extended to a $\mathbb{Z}$-basis of $\text{Sym}^2(\mathbb{Z}^n)$. This describes the corresponding singularities of the compactification.
- **Theorem**: If $\mathcal{V} = \{v_1, \ldots, v_m\}$ is a configuration of shortest vectors in dimension $n$ such that $\text{rk}(\mathcal{V}) = r$ with $r \in \{n, n + 1, n + 2\}$. Then $m = r$.
- The proof of this is relatively elementary and use simple combinatorial arguments (DS., Hukek, Schürmann, 2015).
- **Conjecture**: The equality $m = r$ also holds if $r \in \{n + 3, n + 4\}$.
- No extension to $T$-spaces.
Enumeration of vector configurations for $r = n + 1$, $r = n + 2$

Suppose we know the configuration of shortest vectors in dimension $n$ of rank $r = n$.

- Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be a short vector configuration with $n$ vectors.
- We search for the vectors $v$ such that $\mathcal{W} = \mathcal{V} \cup \{v\}$ is a vector configuration.
- We can assume that $\mathcal{V}$ has maximum determinant in the $n + 1$ subvector configurations with $n$ vectors of $\mathcal{W}$. Thus
  \[
  |\det(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n, v)| \leq |\det(v_1, \ldots, v_n)|
  \]
  for $1 \leq i \leq n$.
- The above inequalities determine a $n$-dim. polytope.
- We enumerate all the integer points by exhaustive enumeration.
- We then check for realizability of the vector families.

For rank $r = n + 2$, we proceed similarly.
Enumeration of vector configurations for $r > n + 2$

We assume that we know all the realizable vector configurations of rank $r - 1$ and $r - 2$.

- We enumerate all pairs $(V, W)$ with $V \subset W$, $rk(V) = r - 2$ and $rk(W) = r - 1$.
- If we have a configuration of rank $r$, then it contains a configuration $V$ of rank $r - 2$ and dimension $n$ which is contained in two configurations $W_1$ and $W_2$ of rank $r - 2$ such that $V \subset W_1$ and $V \subset W_2$.
- So, we combine previous enumeration and obtain a set of configurations $W_1 \cup W_2$
- We check for each of them if there exist a realizable vector configuration $W$ such that $W_1 \cup W_2 \subset W$ and $rk(W) = r$. 
Enumerating the configurations of rank $r = n$

- This is in general a very hard problem with no satisfying solution.
- It does not seem possible to use the polyhedral structure in order to enumerate them.
- The only known upper bound on the possible determinant of realizable configurations $V$ (Keller, Martinet & Schürmann, 2012) is

\[ |\det(V)| \leq \left\lfloor \gamma_n^{n/2} \right\rfloor \]

with $\gamma_n$ the Hermite constant in dimension $n$.
- This bound is tight up to dimension 8.
- For dimension 9 and 10 the bound combined with known upper bound on $\gamma_n$ gives 30 and 59 as upper bound.
- Any improvement, especially not using $\gamma_n$, would be very useful.
The case of prime cyclic lattices

- For a prime $p \in \mathbb{N}$ we consider a lattice $L$ spanned by $e_1 = (1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$ and $e_{n+1} = \frac{1}{p}(a_1, \ldots, a_n)$, $a_i \in \mathbb{Z}$

such that $(e_1, \ldots, e_n)$ is the configuration of shortest vectors of a lattice.

- By standard reductions, we can assume that
  - $a_1 \leq a_2 \leq \cdots \leq a_n$.
  - $1 \leq a_i \leq \lfloor p/2 \rfloor$.
  - $(a_1, \ldots, a_n)$ is lexicographically minimal for the action of $(\mathbb{Z}_p)^*$.  

- With above restrictions, the families of vectors can be enumerated via a tree search.
We want to enumerate the configurations of shortest vectors of index $N$.

For a prime index we do the enumeration of all possibilities and check for each of them.

For an index $N = p_1 \times p_2 \times \cdots \times p_m$ we do following:
- First the enumeration for $N_2 = p_1 \times \cdots \times p_{m-1}$.
- For each realizable configuration of index $N_2$ we compute the stabilizer.
- Then we enumerate the overlattices up to the stabilizer action.
- And we check realizability for each of them.

So for $n = 10$ we have to consider up to index 59.
- This gives 17 prime numbers to consider with a maximal number of cases 16301164 for $p = 59$.
- One very complicated case of $49 = 7^2$.
- It would have helped so much to have better bounds on $\gamma_{10}$!
 Enumeration results for $n \leq 11$

Known number of orbits of cones in the perfect cone decomposition for rank $r \leq 12$ and dimension at most 11.

<table>
<thead>
<tr>
<th>$d \setminus r$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>16</td>
<td>23</td>
<td>25</td>
<td>23</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>10</td>
<td>28</td>
<td>71</td>
<td>162</td>
<td>329</td>
<td>589</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>28</td>
<td>115</td>
<td>467</td>
<td>1882</td>
<td>7375</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>13$^a$</td>
<td>106</td>
<td>783</td>
<td>6167</td>
<td>50645</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td>44$^b$</td>
<td>759</td>
<td>13437</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td>283</td>
<td>16062</td>
<td>?</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>6674$^c$</td>
<td>?</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$a$: Zahareva & Martinet, $b$: Keller, Martinet & Schürmann.
others: Grushevsky, Hulek, Tommasi, DS, 2017.
$c$: Partial enumeration, done only up to index 44 with highest realizable index of 32.
Known enumeration results for $n = 12$

- In dimension 12 the combinatorial explosion for configuration of shortest vectors really takes place.
- Up to index 30 we found:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>56</td>
<td>5</td>
<td>22</td>
</tr>
<tr>
<td>7</td>
<td>62</td>
<td>8</td>
<td>501</td>
</tr>
<tr>
<td>10</td>
<td>685</td>
<td>11</td>
<td>397</td>
</tr>
<tr>
<td>13</td>
<td>876</td>
<td>14</td>
<td>3012</td>
</tr>
<tr>
<td>16</td>
<td>8973</td>
<td>17</td>
<td>3173</td>
</tr>
<tr>
<td>19</td>
<td>5369</td>
<td>20</td>
<td>23072</td>
</tr>
<tr>
<td>22</td>
<td>23096</td>
<td>23</td>
<td>12393</td>
</tr>
<tr>
<td>25</td>
<td>19843</td>
<td>26</td>
<td>42627</td>
</tr>
<tr>
<td>28</td>
<td>77019</td>
<td>29</td>
<td>23629</td>
</tr>
</tbody>
</table>

Total: 454576 configurations so far.
Consequences

- **Thm:** With the exception of the cone of the root lattice $D_4$, every cone in the perfect cone decomposition of dimension at most 10 is basic.
- Starting from dimension 11 there are configurations of shortest vectors which are orientable in the sense of homology.
- In dimension 12 there is a configuration of shortest vectors whose orbit under $GL_{12}(\mathbb{Z})$ splits in two orbits under $SL_{12}(\mathbb{Z})$.
- **Conj.** The maximum index in dimension $n \geq 8$ is $2^{n-5}$.
- **Conj.** A configuration of shortest vectors of rank $r = n$ can be extended to a $\mathbb{Z}$-basis of $\text{Sym}^2(\mathbb{Z}^n)$.
- **Conj.** There is a configuration of shortest vectors of a $n$-dimensional lattice of rank $r = n$ with trivial stabilizer (smallest known size is 4).

THANK YOU